Comparing Infinities

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1 Introduction

In our first year mathematics course when we were learning to count infinities for the first time, in a chat, one of my friends asked something like,

> Okay, I understand that the idea is to compare infinities using bijections. But, given any two unknown infinities, how can we make a guess about whether we should start searching for a bijection or not? What's the intuition behind it?

At first, the question seemed very confusing since on my introductory days, I thought infinities must have to do something with distribution densities. But, I could finally rest that thought after I was fully convinced that the rationals are countable. Soon, I started searching for other intuitive ways of commenting on whether two infinities are equal or not without constructing any bijections. Soon, I found something which seemed pretty convincing to me although I never shared it with anyone because I found it extremely difficult to communicate that idea.

But, just a few days ago, somebody at Mathematics Stack Exchange asked a similar question, and I finally tried to pen that idea down. This is the question and my answer to it that I am talking about. Here, when I saw that my answer was of some help to the OP, I decided to really try to give a better idea of this approach in details using more examples.

But, before going any further, there are four things I would like to mention. First, from now on, I will proceed under the assumption that the reader is familiar with the basics of comparing infinities. Secondly, although the question shared is asked using the $\aleph$ notations, I will stick to avoiding them to keep things as simple as possible. Third, nothing in this article is rigorous- I don't even know whether the idea itself is a rigorous one, so we will mostly be talking in vague non-mathematical terms. And fourth, and the most important one, this idea is completely original and that's why I have some doubts about its validity- however, until now, I have not been able to cook up any scenario where this intuition doesn't work [1].

[1] more about this in the last paragraph

2 Motivation Of The Approach

Suppose you want to get to the infinity of the naturals (which from now on will be represented as $\mathbb{N}$- technically, it's $\aleph_0$), and all you have with you is a finite cardinality. In other words, you only have a finite set, say $\mathbb{A}$, of let's say (some finite) n number of elements, and you wish to create $\mathbb{N}$ from that set. The question is, can you do that using $\mathbb{A}$ only **a finite** number of times? Well, let's try.

Let the set $\mathbb{A}$ be of n elements. Can we add (a finite) m more elements to make it $\mathbb{N}$? Of course no, because that will only give us at most $m+n$ elements. Let's take two sets, $\mathbb{A}_1$ and $\mathbb{A}_2$ of n elements each. If we add them up (i.e., take the union, $\mathbb{A}_1 \cup \mathbb{A}_2$), the maximum we can achieve is $2n$ elements. And everybody knows that there are more natural numbers than $2n$ for any finite n. And note that this 2 isn't even that important- we could have taken (a finite) m such sets together and made room for only at most mn number of elements which is still infinitely smaller than $\mathbb{N}$.
Well, maybe *adding* (actually, union) is too basic an operation to take us to $\mathbb{N}$. Let's take the cross product of (a finite) m such finite sets (i.e., $\mathbb{A} \times \mathbb{A} \times \mathbb{A} \ldots m$ times). Can this ever give $\mathbb{N}$? Unfortunately no, since this process can only give at most n^m elements.
What about power sets? If we take a finite set $\mathbb{A}$, can $\mathcal{P}(\mathbb{A})$ ever be finite? What about $\mathcal{P}(\mathcal{P}(\mathbb{A}))$? What about consider the m-th power set for a finite m? Unfortunately, no once again, since this process can again yield at most $2^{2^{\cdot^{\cdot^{n}}}}$ elements.
By now, you must have been convinced that there is no operation (that does not inherently have *an infinity* at its heart) which when applied finite times on a finite cardinality, gives us $\mathbb{N}$. In other words, if we want to reach infinity from a finite set, we have to apply any such operation more than finite number of times.

It is this idea that leads us to the next part.

3 Searching A Similar Pattern In Infinities

Now that we have an idea of a structure of *the leap* from a finite set to an infinite set, let us consider some examples to see if we can find a similar pattern for infinite sets.

3.1 Infinite Subsets of Naturals

Let us consider the set of even numbers

$$\mathbb{E} = \{2, 4, 6 \ldots\}$$

Let's say, we want to know which one of $\mathbb{N}$ or $\mathbb{E}$ is bigger. It is clear that $\mathbb{E}$ can be embedded in $\mathbb{N}$ exactly twice. So, in some sense (no matter how non-mathematical it is), we can say (and we will use these kind of notations further) that

$$\mathbb{N} = 2 \cdot \mathbb{E}$$

So, $\mathbb{N}$ is just $\mathbb{E}$ taken twice. Under this light, doesn't it make sense to guess that the infinities are equal (doesn't it make sense to *feel* that an infinity taken twice, or only a finite number of times for that matter, doesn't change much)? Indeed that's true since we have the bijection $f : \mathbb{N} \to \mathbb{E}$ given by

$$f(n) = 2n$$

which proves our intuition.

Note that $\mathbb{E}$ is not special. We could have made a similar embedding guess for the set of odd numbers, or the set of all naturals divisible by 3, or all naturals of the form $3n + 1$, or the set of all naturals of the form $Gn + R$ where $G > R$ for any G and R.
So, if a set $\mathbb{S}$ is such that

$$\mathbb{N} = k \cdot \mathbb{S}$$

or

$$\mathbb{S} = k \cdot \mathbb{N}$$

for some integer k, (where the $(\cdot)$ representing multiplication is not rigorous, but intuitively clear using the idea of embedding), then $\mathbb{S}$ is the same infinity as $\mathbb{N}$.
So, finite number of $\mathbb{N}$-infinite sets taken together will always give $\mathbb{N}$.

3.2 Cartesian Products

Now that we've tried *adding* sets, what about trying to multiply them? Let's try to compare $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. What does our intuition say? We will again go non-mathematical to say

$$\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$$

i.e., we will literally look at it as $\mathbb{N}$ multiplied by itself twice. Now, let's think about it. Can an infinity multiplied by itself twice (or only a finite number of times) give a bigger infinity? It's the same infinity multiplied by itself only a finite number of times- how can it expand? Doesn't it make sense to guess that they are equal? Well, indeed they are, and the bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is given by

$$f(m, n) = m + \frac{1}{2}(m + n - 2)(m + n - 1)$$

Again, there's nothing special with 2. As we can clearly understand now, $\mathbb{N}^m$ will be the same infinity as that of $\mathbb{N}$ for any **finite** m.

Just keep a small note here (because I don't want another section for it) that since the integers satisfy $\mathbb{Z} = 2 \cdot \mathbb{N}$ (since they are just the naturals, and the negative naturals) and the rationals are of the form $\frac{a}{b}$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}$ which means $\mathbb{Q} = \mathbb{Z} \times \mathbb{N} = \mathbb{N} \times \mathbb{N}$, we have $\mathbb{Q} = \mathbb{Z} = \mathbb{N}$.

Now that we've tried taking $\mathbb{N}$ finite times, let's see what we can cook up with $\mathbb{N}$ crossed with itself $\mathbb{N}$-infinite number of times. So, we are interested in the behaviour of

$$\mathbb{N}^{\mathbb{N}} = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \dots$$

Now is the first time when it looks like we have a break in our trend for finite cardinalities. I don't know whether there's a way to see through it, but we know that $\mathbb{N}^{\mathbb{N}}$ is uncountable, i.e., it is a strictly bigger infinity than that of $\mathbb{N}$ (a very beautiful proof uses Cantor's Diagonalisation Argument). We will use this observation to build our intuition. We can see that although $\mathbb{N}$ multiplied by itself a finite number of times isn't enough to give a bigger infinity than that of $\mathbb{N}$, when we perform this multiplication $\mathbb{N}$-infinite number of times, it leaps to a bigger infinity- *as if, an infinity taken infinite number of times adds another dimension to it, and this dimension gives it the required jump to a bigger infinity.*

It is this intuition that we will again use in the next example.

3.3 Power Sets

One of the most important results in this field is that the power set of any infinity is bigger than the infinity. As of now, we will only think of the the power set of $\mathbb{N}$, and represent it by $2^{\mathbb{N}}$.

Well, think of **all** the subsets of $\mathbb{N}$. What do they contain? They contain the empty set, all the one element subsets, all the two element subsets, ..., all the $(\mathbb{N} - 1)$ element subsets (by which I mean, all subsets of the form $\mathbb{N} \backslash \{n_0\}$ for some $n_0 \in \mathbb{N}$), and $\mathbb{N}$. Forget about the the empty set (we can anyways leave a finite amount out without changing the infinity) and think of the others.
Consider the set of all one element subsets of $\mathbb{N}$ (denote it by $\mathbb{N}_1$) and note that it is the same as $\mathbb{N}$ (as it is just the elements of $\mathbb{N}$ in curly brackets) since we have the bijection $f : \mathbb{N} \to \mathbb{N}_1$ given by

$$f(n) = \{n\}$$

Again, consider the set of all two element subsets of $\mathbb{N}$ (denote it by $\mathbb{N}_2$) and note that it is again the same as $\mathbb{N}$ (since it is just the elements of $\mathbb{N}$ used twice) due to the bijection

$$f(m, n) = \left\{ m + \frac{1}{2}(m + n - 2)(m + n - 1) \right\}$$

and so on..
So, if we consider a row of one element subsets stacked on top of a row of two

element subsets stacked on a row of three element subsets... (all of which are equal to $\mathbb{N}$), we will see that $2^{\mathbb{N}}$ is basically $\mathbb{N}$ stacked on top of itself $\mathbb{N}$ times. In other words, it's again $\mathbb{N}$ taken $\mathbb{N}$ times, and hence the required *leap to a bigger dimension*.

In case you want a proof, you can find one here.

A fun exercise is to try our method to see which one of $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ is bigger.
Hint: Note that any (finite) n element subset of $\mathbb{N}$ (which is an element of $2^{\mathbb{N}}$) appears in $\mathbb{N}^{\mathbb{N}}$ exactly ($n!$) times because of the order built in the elements of $\mathbb{N}^{\mathbb{N}}$.
Some ideas about the bijection can be found here.

4 Finalising The Approach

With reference to what we have established so far, let us restate what we found-

> An infinite set $\mathbb{A}$ taken less than $\mathbb{A}$ times can never give you a bigger infinity than that of $\mathbb{A}$. In other words, if you have two infinities $\mathbb{A}$ and $\mathbb{B}$, and you can find some way of embedding $\mathbb{A}$ within $\mathbb{B}$ using $\mathbb{A}$ at most less than $\mathbb{A}$ times, then you can be sure that $\mathbb{B}$ is the same infinity as that of $\mathbb{A}$. So, to reach a greater infinity, you need to use $\mathbb{A}$, at least $\mathbb{A}$ times.

This part about using $\mathbb{A}$ at least $\mathbb{A}$ times, is what I would like to call, "adding a dimension of infinity" (this has nothing to do with our strict mathematical definition of a dimension, but the word really captures this *leap to a bigger infinity*).

Let us look at some more examples.

5 Testing The Idea

Notice that we were mostly discussing about $\mathbb{N}$ till now. So, we will see whether our idea works for other infinities as well.

5.1 The Reals and Intervals

Since we saw that $\mathbb{Z}$ and $\mathbb{Q}$ were the same as $\mathbb{N}$, it makes sense to randomly guess that maybe $\mathbb{R}$ will also follow the trend.

So, let's try embedding $\mathbb{N}$ in $\mathbb{R}$. So, let us have the set $\mathbb{R}$ and let us cover all the integers- that's one set of $\mathbb{N}$ (since we know that $\mathbb{N} = \mathbb{Z}$). So, we still have a lot of $\mathbb{R}$ left (precisely, $\mathbb{R}\backslash\mathbb{Z}$). How can we cover that up? Let's try covering all numbers of the form $a.1$ for integer a (i.e., numbers of the form

$1.1, 2.1, 3.1\ldots$). Clearly, that's one more set of $\mathbb{N}$ (as they also are just integers with a .1 at the back). But still, a lot of $\mathbb{R}$ is left. So, let's try covering all the numbers of the form $a.2$, then $a.3$, and so on to $a.9$. But still, a lot is left. Okay, cover up all $a.11$, $a.12$,... Now, it does start appearing that our guess was wrong. Due to the continuity of $\mathbb{R}$, it does seem we will never be able to cover them up using $\mathbb{N}$. In other words, there is no k such that

$$\mathbb{R} = k \cdot \mathbb{N}$$

or, no matter how many $\mathbb{N}$s you put in, there will still be a lot of $\mathbb{R}$ remaining.

Okay, so maybe $\mathbb{R} \neq \mathbb{N}$, but then what is it? Is it equal to some other infinities that we know? Let's see.
What do we know about the reals? Aren't they just numbers that look like $A.B$ where A is a finite *natural chunk* (chunks of natural numbers), and B is another (finite or infinite) natural chunk. So, the reals are just a set of all (finite or infinite) natural chunks with all possibilities of placing a decimal in between. Have we seen any other set of natural chunks before? What about $\mathbb{N}^{\mathbb{N}}$?
As you may have noticed, the placement of the decimal still creates some problems. Then, just fix it.
Let us consider only the reals in the interval $\mathbb{I} = [0, 1)$ which look like $0.abcd...$ or $0.C$ where C is a (finite or infinite) natural chunk. If we just forget the "0." for a second (since that part is constant), it would be evident that $\mathbb{I} = \mathbb{N}^{\mathbb{N}}$. But, we of course have $\mathbb{I} \leq \mathbb{R}$ which means $\mathbb{R} \geq \mathbb{N}^{\mathbb{N}} > \mathbb{N}$.
Now, notice that $\mathbb{I}$ is just an unit length interval which contains the first point, but doesn't contain the last. So, consider tiling (same as embedding, but "tiling" seems to be a better word for continuous cases) $\mathbb{R}$ with $\mathbb{I}$. Isn't it clearly visible that you will need exactly $\mathbb{Z}$ (=$\mathbb{N}$) such tiles? And since $\mathbb{Z}$ is less than $\mathbb{R}$ (as we just established), isn't it evident that $\mathbb{R} = \mathbb{N}^{\mathbb{N}}$ (since it's $\mathbb{R}$ taken $\mathbb{N}(< \mathbb{R})$ times)? So, there you have it!

But what about intervals other than $[0, 1)$? We will only talk about the intervals $\mathbb{I}_x = [0, x)$ (since $[a, b)$ can be seen as $[0, b - a)$ and also, whether the endpoints are included or not doesn't really matter since adding or taking away a finite number of elements doesn't change anything).
If $x > 1$, then notice that $\mathbb{I}_{\lfloor x \rfloor} \leq \mathbb{I}_x \leq \mathbb{I}_{\lceil x \rceil}$ and since both $\mathbb{I}_{\lfloor x \rfloor}$ and $\mathbb{I}_{\lceil x \rceil}$ are of the form $k \cdot \mathbb{I}$ $(= \mathbb{I})$ for some k, we must have $\mathbb{I}_x = \mathbb{I}$.
And if $x \in (0, 1)$, we will use intervals of the form $\mathbb{I}_{\frac{1}{10^n}}$ to use a similar argument as above.
And for intervals containing a negative real number, just shift it to make it positive.
(Note that the set notations were often used to denote cardinality)

The two main equivalences that we discussed in this section are that of $\mathbb{N}^{\mathbb{N}}$ with $\mathbb{I}$ and of $\mathbb{I}$ with $\mathbb{R}$. For the first one, the bijection is quite clear. For the second, the required bijection is

$$f(x) = \tan(\pi(x - 0.5))$$

5.2 Products and Power Sets of Reals

We have seen about the infinity of the products and power sets of naturals before. So, now let's look at the reals and observe the similarities and the differences.

Let us have a look at the product

$$\mathbb{R}^k = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \ k \text{ times}$$

This is of course same as $\mathbb{R}$ for the same reason why $\mathbb{N}^k = \mathbb{N}$.

But, what about taking the product $\mathbb{N}$-infinite number of times, i.e., we want to look at

$$\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots$$

Last time when $\mathbb{N}$ was taken $\mathbb{N}$ times, we got that *leap* that we were talking about. But, isn't this time it's different? Last time, $\mathbb{N}$ was taken $\mathbb{N}$ times; this time, $\mathbb{R}$ is taken $\mathbb{N}$ ($< \mathbb{R}$) times. So, from the patterns we have been observing till now, doesn't it seem that $\mathbb{R}$ will NOT have that required leap, and so $\mathbb{R}^{\mathbb{N}} = \mathbb{R}$? That's of course true and ideas for the corresponding bijection can be found here.

Then, what about power sets? Under the light of how we discussed products, it seems like the reals will again not have that required leap which kicked in in the case of $\mathbb{N}$. But, that's just what it seems.
Consider the set $\mathcal{P}(\mathbb{R})$. Of course, it has got all n element subsets for any $n \in \mathbb{N}$, i.e., $\mathbb{N}$ copies of $\mathbb{N}$- that gives one copy of $\mathbb{R}$. But, that's just $\mathbb{R}$. We are more interested in the other subsets.
Now, consider a positive x and look at the intervals $\mathbb{I}_x$ as we defined above. Since we know that $\mathbb{I}_x = \mathbb{R}$, can you see that you have $\mathbb{R}^+(= \mathbb{R}$ since $2 \cdot \mathbb{R}^+ = \mathbb{R})$ copies of $\mathbb{R}$ in $2^{\mathbb{R}}$? So, there you have the verification of $\mathcal{P}(\mathbb{R}) = 2^{\mathbb{R}} > \mathbb{R}$!
Note that last time we said, we have $\mathbb{N}$ stacks of $\mathbb{N}$. This time, "stack" doesn't look to be a promising term since it makes more sense over natural indexes. But, the intuition is still clear. You have **all** $\mathbb{I}_x$ for x varying over the reals. This is what gives you $\mathbb{R}$ copies of $\mathbb{R}$, and hence gives you that added dimension.

In general, it's not hard to see (after these two examples are discussed) that if you have any general infinity $\mathbb{A}$, we will have at least $\mathbb{A}$ copies of $\mathbb{A}$ in $\mathcal{P}(\mathbb{A})$.

A fun exercise is to compare $\mathbb{R}^{\mathbb{N}}$ with $2^{\mathbb{R}}$ and contrast that with how we compared $\mathbb{N}^{\mathbb{N}}$ with $2^{\mathbb{N}}$.

5.3 A Line And A Plane

One last example we will consider is the infinity of points on a line segment and a square.

This is really nothing new. We can think of a square as a subset of $\mathbb{R}^2$ with any two adjacent sides being the two axes (that is, we are shifting the square so that one of its vertices lie on the origin). Also, a line segment is just an interval of $\mathbb{R}$.
We have already established that an interval and $\mathbb{R}$ are of the same infinity. We can use a similar tiling process to argue that a square and $\mathbb{R}^2$ are of the same infinity.
And, since we know that $\mathbb{R}$ and $\mathbb{R}^2$ are the same infinities, it is quite clear that the infinity of the line segment and the plane are same.

Note that the square is nothing special and any shape in $\mathbb{R}^2$ would have done a similar job (use squares to bound the figure like we did in the case of intervals). Also, the idea works for any shape in $\mathbb{R}^n$ for any n. That is, the same tiling and bounding algorithm can be applied to show that the infinity of points inside a cone and a circle is the same; not just that, the infinity of points in the volume and the surface of any figure in any dimension is equal. In other words, you may go as wild as you want to- let's say you consider your favourite shape in $\mathbb{R}^{100}$ and your friend's favourite shape in $\mathbb{R}^{500}$. The points in their volume make the same infinity.

It is this idea (of the infinity of line and plane being equal) that leads us to Space Filling Curves.

6 A Strange Consequence

Now, we will talk about another strange idea again in vague terms. It's not really an idea- just a small comment.
If you would have pondered on our discussions about "adding a dimension", you might have been tempted to think it as "adding **one** dimension" or "going to **the next** infinity". That is, life seems to have been easier if we could have said that "$\mathbb{R}$ is just one dimension added to $\mathbb{N}$", or "$\mathcal{P}(\mathbb{A})$ is one dimension added to $\mathbb{A}$". But, that's not true. And that leads us to one of the greatest and most studied questions in this field- The Continuum Hypothesis. I won't go any deeper into it, but just felt like sharing how far this idea can lead us to.

In our terms, the Continuum hypothesis translates to something like

> Considering the *change in dimension* that we were talking about, can we associate numbers with them and do math with them? In other words, is it okay to say that there is exactly one *leap* from $\mathbb{N}$ to $\mathbb{R}(= 2^{\mathbb{N}})$ or are there more in between?

If the answer to this question would have been yes (which it isn't), things would have been much easier since we could have literally associated a number with each dimension of infinity, and hence, a number denoting how many leaps we have in between. But since the Continuum Hypothesis is undecidable, we can never really *count* infinities- we can only *compare* them.

7 Drawbacks

As I have already mentioned in the introduction, this idea is completely original and hence, I have some doubts about its validity. I mean, the embedding, tiling and bounding, etc. look quite intuitive and I'm quite confident that they are true. But, I have some doubts regarding the *leap* or the *added dimension* that we were talking about. However, no matter how hard I tried, I couldn't construct a scenario where this intuition lead to the wrong answer. But, as it turns out, there may be cases where these ideas don't seem to conclude anything significant.

One example that is instantly evident is that of $\mathbb{P}$ denoting the set of all primes. Since the primes don't follow any particular order and are almost randomly living inside $\mathbb{N}$, it seems, there's no way to argue that $\mathbb{P}$ can be embedded inside $\mathbb{N}$. What's even more unfortunate is that we can't even provide bounds for $\mathbb{P}$ like we did for the intervals. We can provide an upper bound for sure given by

$$\mathbb{P} \leq \mathbb{S}_1 \cup \mathbb{S}_2$$

where

$$\mathbb{S}_1 = \{x : x = 6n + 1, n \in \mathbb{N}\}$$

and

$$\mathbb{S}_2 = \{x : x = 6n - 1, n \in \mathbb{N}\}$$

while

$$6 \cdot \mathbb{S}_1 = 6 \cdot \mathbb{S}_2 = \mathbb{N}$$

which means $\mathbb{P} \leq \mathbb{N}$.
But, as far as I know, there is no nice form which can be embedded easily in $\mathbb{N}$ as well as be shown to be a subset of $\mathbb{P}$. So, unfortunately, there is no way for us to establish this simple fact that the infinity of $\mathbb{P}$ and $\mathbb{N}$ are equal.

Once we have this example of $\mathbb{P}$, we can see that the same scenario repeats for any random infinite subset of $\mathbb{N}$. While considering bijections make it immediately evident that $\mathbb{N}$ is the same infinity as its infinite subsets, our approach doesn't seem like doing it that easily. In other words, our ideas cannot give a detailed intuition of why $\mathbb{N}$ is the smallest infinity. Maybe, assuming $\mathbb{N}$ to be the smallest infinity can rest some of our difficulties.

Another instance where this idea doesn't work is when we want to compare

two completely different sets. For example, let's say we want to compare $\mathbb{N}$ and the set of infinite mangoes. There cannot be any embedding technique that can provide us an equality between the two. This is because these two sets live in completely different realms of reality (although abstractly they are basically the same sets).
A mathematical equivalent of this is when we have two completely different spaces to compare. Though I don't have any concrete examples to discuss here, but I guess, it's not difficult to understand that if we have two absolutely different notions, it's not possible for us to compare them without constructing bijections.

However, even considering these super extreme cases where this intuition may fail, it still seems that this idea is quite a feasible option to quickly compare two different infinities.