Unfolding Pythagorean Triples from the Unit Circle

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Introduction

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These solutions are called *primitive solutions*.

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If (u, v) is another rational solution of (2), we can join these two points and get the straight line

$$y = \frac{v}{u+1}(x+1)$$

Rational Hunt continues

This straight line has a *rational* slope

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$$y = t(x+1)$$

Plugging this into (2), we get

$$x^2 + t^2(x+1)^2 = 1, \qquad x = -1, \ \frac{1-t^2}{1+t^2}$$

Rational treasure found

So, $y = t(x+1)$ meets the unit circle at the *rational* point

$$(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \tag{3}$$

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So, by drawing **all** such lines of rational slope, we can be sure that we have accounted for **every** rational point on our circle!

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Parity Analysis

Integer Hunt continues

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So, there must be some integer k such that

$$kX = m^2 - n^2, \qquad kY = 2mn, \qquad kZ = m^2 + n^2$$

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Now,

$$\begin{aligned} & k \mid m^2 - n^2 \quad \text{and} \quad k \mid m^2 + n^2 \\ \Rightarrow \quad & k \mid 2m^2 \qquad\quad \text{and} \quad k \mid 2n^2 \\ \Rightarrow \quad & k \mid 2 \end{aligned}$$

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which is a contradiction.

Hence, we must have $k = 1$.

Final Parametrization

Plugging in $k = 1$, we get the final solution as

$$(X, Y, Z) = (m^2 - n^2, 2mn, m^2 + n^2)$$

for coprime m, n, one odd and the other even.

Jargon

1. Rational points are those of the form (p, q) with $p, q \in \mathbb{Q}$.
2. Rational lines are those of the form $ax + by + c = 0$ with $a, b, c \in \mathbb{Q}$.
3. Rational conics are those of the form $ax^2 + bxy + cy^2 + dx + fy + g = 0$ with $a, b, c, d, f, g \in \mathbb{Q}$.

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Some observations:

1. A line passing through two rational points is a rational line.
2. Two rational lines intersect at a rational point.
3. A rational conic and a rational line (and hence, two rational conics) **may not** intersect at rational points.
   For example, $y = x^2 + 1$ and $y = x + 2$ intersects at $x = \frac{1 \pm \sqrt{5}}{2}$.

An Interesting Question

We want to address the question of whether we can generalize these tricks to find integer solutions to any equation of the form

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Let us note that, if $n = s^2 + t^2$, then

$$X^2 + Y^2 = (s^2 + t^2)Z^2 = (sZ)^2 + (tZ)^2$$

and hence, $(X, Y) = (sZ, tZ)$ is a solution.

The (not so) Interesting Expression

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$$\begin{aligned} x &= x_0 np^2 - 2qnz_0 p + 2q^2 y_0 \\ y &= y_0 np^2 - 2qnz_0 p + 2q^2 x_0 \\ z &= z_0 np^2 - 2qx_0 p - 2qy_0 p + 2q^2 z_0 \end{aligned}$$

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We basically want to find at least one rational point on the ellipse

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and this turns out to be much more difficult than the last one.

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Theorem (Legendre)

Let a, b, c be coprime positive integers. Then, the equation

$$aX^2 + bY^2 = cZ^2$$

has a non-trivial rational solution iff

$$\left(\frac{-bc}{a}\right) = \left(\frac{-ac}{b}\right) = \left(\frac{ab}{c}\right) = 1$$

Jacobi Symbol

Definition (Jacobi Symbol)

The Jacobi Symbol written as $\left(\frac{n}{m}\right)$ is defined for positive odd m as

$$\left(\frac{n}{m}\right) = \left(\frac{n}{p_1}\right)^{a_1} \left(\frac{n}{p_2}\right)^{a_2} \cdots \left(\frac{n}{p_k}\right)^{a_k}$$

where $m = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ and $\left(\frac{n}{p_1}\right)$ is the Legendre symbol.

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Definition (Legendre Symbol)

For an odd prime p and an integer a, we define the Legendre Symbol of a with respect to p as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p} \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \\ 0 & \text{if } a \equiv 0 \pmod{p} \end{cases}$$

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Theorem (Hasse)

A homogeneous quadratic equation in several variables is solvable by integers, not all zero, if and only if it is solvable in real numbers and in p-adic numbers for each prime p.

Fermat's Last Theorem

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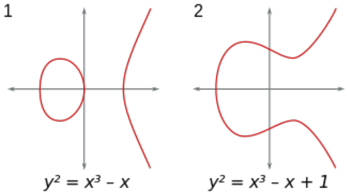
Sadly, we cannot directly use the geometric principle that worked so well for conics because a line generally meets a cubic in three points. And if we have one rational point, we cannot project the cubic onto a line, because each point on the line would then correspond to two points on the curve.

Why Rational points?



Elliptic Curves

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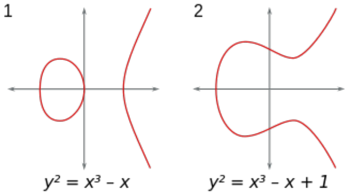
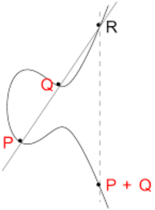


Elliptic Curves



Rational Points on Elliptic Curves

Love Poems

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Theorem (Mordell)

Let C be a non-singular cubic curve with rational coefficients. Then the group Γ of rational points on C is finitely generated.

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Theorem (Nagell-Lutz)

Let

$$y^2 = x^3 + ax^2 + bx + c$$

be a non-singular cubic curve with integer coefficients a, b, c, and let D be the discriminant of the cubic polynomial

$$D = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2$$

Let $P = (x, y)$ be a rational point of finite order. Then x and y are integers, and either $y = 0$, or $y | D$.

Thank You!

References

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2. The Desmos Animation

3. Square modulo 4 animation by Satvik Saha

4. Legendre's Theorem